QUANTUM CREATION OF A UNIVERSE WITH QUASI-HYPERBOLIC SPATIAL SECTIONS*

Helio V. Fagundes and Teofilo Vargas
Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona, 145, 01405-900 São Paulo, SP
Brazil

February 7, 2008

Abstract

We calculate the probability of creation of a universe with space topology $S^1 \times T_g$, where S^1 is the circle and T_g is a compact hyperbolic surface of genus $g \geq 2$. We use the method of path integrals as applied to quantum cosmology.

1 Introduction

In recent years there has been much interest in quantum cosmology, and the question of topology has been added to the discussion. Which is the most probable topology of the universe? It is well known that there exist a lot more hyperbolic manifolds than elliptic and flat ones. So it is interesting to study the hyperbolic manifolds in quantum cosmology.

Here we shall study the probability of creation of a universe with the space topology $S^1 \times T_g$. The anisotropic minisuperspace path integral reduces to a single ordinary integration over the lapse, then this integral is evaluated by the lapse method. We will find that information about the space topology is encoded in the measure of integration as well as in the probability of creation.

*Contribution to the XXIV National Meeting on the Physics of Particles and Fields, held in Caxambu, MG, Brazil, in October 2003.

The wave function of the universe as proposed by Hartle-Hawking (HH) is given by the path integral [1]

$$\Psi(h_{ij}) = \int D(g_{\mu\nu}) \exp[-S_E(g_{\mu\nu})] \tag{1}$$

where h_{ij} is the metric on the space section, and S_E is the Euclidean action of the gravitational field with a cosmological constant

$$S_E = -\frac{1}{16\pi G}\int_\Omega d^4x\, g^{1/2}\,(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\Omega} d^3x\, h^{1/2}\, K\,. \tag{2}$$

The Euclideanized line element of the Bianchi type III universe is given by

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 dr^2 + b(\tau)^2 d\Omega_g^2, \tag{3}$$

where the coordinate r has a period 2π, and the metric of the compact hyperbolic surface of genus g, T_g, is $d\Omega_g^2 = \rho^2 + \sinh^2\rho d\phi^2$.

The volume of the space section with the topology $S^1 \times T_g$ is [2]

$$V = 8\pi(g-1)ab^2\,, \tag{4}$$

so the Euclidean Einstein-Hilbert action is

$$S_E = \frac{(g-1)}{2G}\int d\tau\left[-\frac{2\dot{a}\dot{b}b}{N} - \frac{a\dot{b}^2}{N} + Na + Nab^2\Lambda\right]. \tag{5}$$

Thus the information about the topology of the spatial compact section is encoded in the action. In the gauge in which $\dot{N}=0$, the integration of the field equation gives

$$\dot{b}^2 + \frac{\Lambda}{3}N^2b^2 + N^2 - \frac{C}{b} = 0\,. \tag{6}$$

Putting the integration constant C equal to zero we see that the cosmological constant should be negative, and integrating once more, we obtain

$$b(\tau) = \sqrt{\frac{3}{|\Lambda|}}\cosh\left[\sqrt{\frac{|\Lambda|}{3}}N\tau\right]\;,\; a(\tau) = \sqrt{\frac{3}{|\Lambda|}}\sinh\left[\sqrt{\frac{|\Lambda|}{3}}N\tau\right]\;. \tag{7}$$

For arbitrary C the solution of equation (6) is expressed in terms of elliptic funtions. The general solution for arbitrary C as well as for $\Lambda < 0$ is under our investigation. Here we are interested in the approximate evaluation of the path integral (1) by the lapse method.

2 The lapse method and evaluation of path integral

The lapse method, as suggested be Halliwell and Hartle [3], consists in separating the anisotropic minisuperspace path integral (1) into an essentially trivial functional integral over the scale factors $a(\tau)$, $b(\tau)$ and a nontrivial ordinary integral over the lapse $N(\tau)$, and then evaluating the propagation amplitude between fixed initial and final values of the scale factors.

Now let us rewrite the minisuperspace action (5) for fixed initial and final values of the scale factors. Rescaling the lapse $N \longrightarrow N/a$, and expressing S_E in terms of the variables b, c, and N, the action is

$$S_E(b, c, N) = \frac{(g-1)}{2G} \int_0^1 d\tau \left[-\frac{\dot{b}\dot{c}}{N} + N + Nb^2\Lambda \right] , \tag{8}$$

where $c = a^2 b$.

The anisotropic minisuperspace quantum propagation amplitude between fixed initial and final values of the scale factors, in the gauge $\dot{N} = 0$, is given by

$$\mathbf{G}(b'', c''|\, b', c') = \int dN \int Db\, Dc\, \exp[-S_E(b, c, N)] , \tag{9}$$

where the functional integrals over b and c satisfy the boundary conditions

$$b(0) = b' \,,\ b(1) = b'' \,,\ c(0) = c' \,,\ c(1) = c'' \,. \tag{10}$$

The lapse funtion N is in general unrestricted and then the propagator $\mathbf{G}$ is a solution of the Wheeler-DeWitt equation at the final state [4].

From the boundary condition we know that the classical solution must be everywhere regular, the boundary term at $\tau = 0$ should vanish, and the quantum 4-metric has a vanishing 3-volume at the bottom. From (7) we see that

$$a(0) = 0 \,,\ \frac{1}{N} \frac{da(0)}{d\tau} = 1 \,, \tag{11}$$

which is consistent with above requirement.

The initial conditions must be consistent with quantum mechanics, which means that one is not attempting to fix too many pieces of initial data or to fix a coordinate and its conjugate momentum simultaneously. Considering the relationship between the velocities and the momenta in adequately chosen new variables, and assuming $\Lambda = 0$, it is shown in [5] that the wave function may be approximately evaluated by

$$\Psi(a, b) \approx \int dN\, \mu(a, b, N) \exp[-S_o(a, b, N)] , \tag{12}$$

where

$$S_o(a,b,N) = \frac{(g-1)}{G}\left[\frac{a^4b^2}{8N^2} - \frac{a^2b^2}{2N} + \frac{N}{2}\right], \tag{13}$$

and the measure of integration is

$$\mu(a,b,N) = \frac{(g-1)}{G} f(a,b) N^{-3/2}\left[1 - \frac{a^2}{2N}\right]^{1/2}. \tag{14}$$

Thus the minisuperspace path integral reduces to a single ordinary integration over the lapse.

Now let us perform a steepest descent analysis of the N integration in (12). The saddle points are the values of N for which $\partial S_o/\partial N = 0$ and are the roots of the cubic equation, with one real (N_1) and a complex-conjugate pair of roots with a negative real part (N_2, N_3). Analysing the asymptotic forms at large a/b and ignoring the prefactor, Halliwell and Louko [5] showed that the path integral over the real Euclidean contour with positive lapse (N_1), is in general divergent when integrated over real Euclidean geometries, for the half-infinite lapse contours along the positive (N_2) and negative (N_3) imaginary axes the semiclassical wave functions are of the form

$$\Psi_\pm(a,b) \approx \exp\left[\frac{(g-1)}{G}\frac{a^2}{8}\right]\exp(\pm iab). \tag{15}$$

These wave functions have a rapidly varying phase and a slowly varying exponential factor, and so corresponds to an ensemble of classical Lorentzian trajectories, weighted by the exponential factor.

Thus the (unnormalized) probability of creation of a universe with the topology taken into consideration is

$$|\Psi_\pm(a,b)|^2 \approx \exp\left[\frac{(g-1)}{G}\frac{a^2}{4}\right]. \tag{16}$$

3 Final Remarks

We considered the probability of creation of a universe with the space topology $S^1 \times T_g$. The anisotropic minisuperspace path integral reduces to a single ordinary integration over the lapse, and we study this integral by the lapse method. We found that the information about the space topology is encoded in the measure of the integration (14) as well as in the probability of creation (16).

Acknowledgment

One of us (T. V.) would like to thank FAPESP for financial support.

References

[1] J. B. Hartle and S.W. Hawking, Phys.Rev. D28 (1983) 2960; J. J. Halliwell, in *Quantum Cosmology and Baby Universes*, eds. S. Coleman, J. B. Hartle, T. Piran and S. Weinberg (World Scientific Singapore, 1991), p. 159.

[2] H. V. Fagundes, Lett. Math. Phys. 6 (1992) 96.

[3] J. J. Halliwell and J. B. Hartle, Phys.Rev. D41 (1990) 1815.

[4] J. J. Halliwell and J. Louko, Phys.Rev. D40 (1989) 1868.

[5] J. J. Halliwell and J. Louko, Phys.Rev. D40 (1989) 4011.